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Quantum Charged Non-Linear Nano-String and Quantum Vacuum

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The classical and quantum dynamic of a non-linear charged vibrating string and its interaction with quantum vacuum field is investigated. Some probability amplitudes for transitions between vacuum field and quantum states of the string are obtained. The effect of non-linearity on some probability amplitudes is investigated and finally the correct equation for string containing the vacuum and radiation reaction field is obtained.

KEY WORDS: nano-string; non-linear; quantum vacuum; radiation reaction.

1. INTRODUCTION

In QED a charged particle in quantum vacuum interacts with the vacuum field and its own field known as radiation reaction. In classical electrodynamics there is only the radiation reaction field that acts on a charged particle in the vacuum. The vacuum and radiation reaction fields have a fluctuation–dissipation connection (Milonni, 1994) and both are required for the consistency of QED. For example, the stability of the ground state, atomic transitions and lamb shift can only be explained by taking into account both fields. If self reaction was alone the atomic ground state would not be stable (Faria *et al.*, 2004; Milonni, 1994). In some cases the self reaction effects can be derived equivalently from the corresponding classical radiation theory (Dalibard *et al.*, 1982). When a quantum mechanical system interacts with the quantum vacuum of electromagnetic field, the coupled Heisenberg equations for both system and field give us the radiation reaction field. For example, it can be shown that the radiation reaction for a charged harmonic oscillator is $\frac{2e^2}{3c^3}$, (Faria *et al.*, 2004; Milonni, 1994). One method for generating coherent states is by interacting a classical current source with the quantized electromagnetic field, where the probability P_n for emission of n photons when

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neither the momenta nor the polarization are observed is a Poissonian distribution (Itzykson, 1985). In this paper,we investigate the quantum dynamics of a non-linear nano-scale charged string and its interaction with the quantum vacuum field. The effect of non-linear term on probability amplitude for some transitions are investigated. In the last section, the radiation reaction field for the quantized vibrating string is derived and it is shown that the correct equation for string contains the vacuum as well as the radiation.

2. QUANTUM NON-LINEAR DYNAMICS OF A NANO VIBRATING CHARGED STRING

Consider a nano-string with length L on x-axis and let's apply the periodic boundary conditions. The mechanical wave function $y(x, t)$ of this string satisfies the following non-linear equation

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = \frac{\gamma}{2} \frac{\partial^2 y}{\partial x^2} \left(\frac{\partial y}{\partial x} \right)^2, \tag{1}$$

where γ and v are constants depending on the string. This equation can be obtained from the following Lagrangian density

$$\pounds = \frac{1}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{1}{2} v^2 \left(\frac{\partial y}{\partial x} \right)^2 - \frac{\gamma}{24} \left(\frac{\partial y}{\partial x} \right)^4, \tag{2}$$

the canonical momentum density corresponding to $y(x, t)$ is

$$\pi_y(x, t) = \frac{\partial \pounds}{\partial(\partial_t y)} = \frac{\partial y(x, t)}{\partial t} \tag{3}$$

and the canonical quantization rule is (Greiner, 1996)

$$[\pi_y(x, t), y(x', t)] = -i\delta(x - x') \tag{4}$$

The Lagrangian density (2) give us the Hamiltonian

$$H_s(t) = \int_0^L dx \left(\frac{\pi_y^2}{2} + \frac{1}{2} v^2 \left(\frac{\partial y}{\partial x} \right)^2 + \frac{1}{24} \gamma \left(\frac{\partial y}{\partial x} \right)^4 \right). \tag{5}$$

By expanding $y(x, t)$ and $\pi_y(x, t)$ in terms of orthogonal periodic functions $e^{\frac{i2\pi nx}{L}}$

$$y(x, t) = \sum_{n=-\infty}^{+\infty} C_n(t) e^{\frac{i2\pi nx}{L}} \qquad \pi_y(x, t) = \sum_{n=-\infty}^{+\infty} \dot{C}_n(t) e^{\frac{i2\pi nx}{L}} \tag{6}$$

and using (4) we find the following commutation relations

$$[C_n(t), \dot{C}_m(t)] = \frac{i}{L} \delta_{n,-m} \qquad [C_n(t), \dot{C}_m{}^{\dagger}(t)] = \frac{i}{L} \delta_{n,m}, \tag{7}$$

because of hermiticty of operators $y(x,t)$ and $\pi_y(x,t)$ we find from (6),

$$C_n^\dagger(t) = C_{-n}(t), \quad \dot{C}_n^\dagger(t) = \dot{C}_{-n}(t), \tag{8}$$

now we rewrite the Hamiltonian (5) in terms of C_n's

$$\begin{aligned} H_s(t) &= H_{1s} + H_{2s} \\ H_{1s} &= \frac{L}{2}\sum_{n=-\infty}^{+\infty} \dot{C}_n(t)\dot{C}_n{}^\dagger + \frac{L}{2}\sum_{n=-\infty}^{+\infty} \omega_n^2 C_n C_n^\dagger \\ H_{2s} &= -\sum_{n,s,t=-\infty}^{+\infty} M_{n,s,t} C_n C_s C_t C_{n+s+t}^\dagger \\ M_{n,s,t} &= \frac{2\pi^4\gamma}{3L^3} nst(n+s+t) \end{aligned} \tag{9}$$

where $\omega_n = \frac{2\pi v n}{L}$. Hamiltonian (9) give the Heisenberg equations for C_j's

$$\begin{aligned} \ddot{C}_j + \omega_j^2 C_j &= \sum_{s,t=-\infty}^{+\infty} L_{s,t} C_s C_t C_{s+t-j}^\dagger \\ L_{s,t} &= -\frac{8\pi^4\gamma j}{3L^4} st(s+t-j) \end{aligned} \tag{10}$$

Let us define the operator $\widetilde{O}(t)$, for any operator O^s in Schrodinger picture as

$$\widetilde{O}(t) = e^{iH_{1s}t} O^s e^{-iH_{1s}t}, \tag{11}$$

so in the special case for the operators $C_n(t)$ we have

$$\widetilde{C}_n(t) = e^{iH_{1s}t} C_n(0) e^{-iH_{1s}t} = B_{-n}^\dagger e^{i\omega_n t} + B_n e^{-i\omega_n t}. \tag{12}$$

where we have used from (8). New operators B_n and $B_{-n}^\dagger$, are annihilation and creation operators of phonons of type $|n\rangle$, i.e, with the wave function $\frac{1}{\sqrt{L}} e^{(\frac{2\pi n}{L}x)}$ and satisfy the following commutation relations

$$[B_n, B_m^\dagger] = \frac{\delta_{n,m}}{2L\omega_n}. \tag{13}$$

A state with j phonons in fock space with the corresponding modes, $s_1, s_2, \ldots, s_j$, is denoted by $|s_1, s_2, \ldots, s_j\rangle$, such that

$$\begin{aligned} B_m|s_1, s_2, \ldots, s_j\rangle &= \frac{1}{\sqrt{2L\omega_m}} \sum_{r=1}^{j} \delta_{m,s_r} |s_1, \ldots, s_{r-1}, s_{r+1}, \ldots, s_j\rangle, \\ B_m^\dagger|s_1, s_2, \ldots, s_j\rangle &= \frac{1}{\sqrt{2L\omega_m}} |m, s_1, \ldots, s_j\rangle, \end{aligned} \tag{14}$$

substituting $C_n(0) = \widetilde{C}_n(0)$ from (12) in (9), gives the Hamiltonian H_s^s in Schrodinger picture

$$\begin{aligned}
H_s^s &= H_{1s}^s + H_{2s}^s \\
H_{1s}^s &= 2L\sum_n \omega_n^2 B_n^\dagger B_n \\
H_{2s}^s &= -6\sum_{n,s,t} M_{n,s,-t} B_n^\dagger B_s^\dagger B_t B_{n+s-t} - 4\sum_{n,s,t} M_{n,s,-t} B_t^\dagger B_s B_n B_{t-s-n} \\
&\quad - \sum_{n,s,t} M_{n,s,t} B_n^\dagger B_s^\dagger B_t^\dagger B_{-n-s-t}^\dagger - 4\sum_{n,s,t} M_{n,s,t} B_n^\dagger B_s^\dagger B_t^\dagger B_{n+s+t} \\
&\quad - \sum_{n,s,t} M_{n,s,t} B_n B_s B_t B_{-n-s-t}
\end{aligned} \tag{15}$$

The eigenvalues of H_s^s up to the first-order perturbation, can be obtained from the following relation

$$\begin{aligned}
E_{s_1,\ldots,s_j} &= \sum_{i=1}^{j} \omega_{s_i} + E_{s_1,\ldots,s_j}^{(1)}, \\
E_{s_1,\ldots,s_j}^{(1)} &= \left\langle s_1,\ldots,s_j \right| H_{2s}^s \left| s_1,\ldots,s_j \right\rangle = \sum_{i=1}^{j}\sum_{r\neq i} \frac{2\gamma\pi^4}{3\omega_{s_i}^2\omega_{s_r}^2 L} s_i^2 s_r^2.
\end{aligned} \tag{16}$$

Let $|\psi(t)\rangle_s$ be the state vector in Schrodinger picture and define $|\widetilde{\psi(t)}\rangle = e^{-iH_{2s}t}|\psi(t)\rangle_s$, then $|\widetilde{\psi(t)}\rangle$ satisfies the time evolution equation

$$i\frac{\partial}{\partial t}|\widetilde{\psi(t)}\rangle = \widetilde{H}_{2s}(t)|\widetilde{\psi(t)}\rangle, \tag{17}$$

and up to the first-order perturbation $|\widetilde{\psi(t)}\rangle$ we have

$$|\widetilde{\psi(t)}\rangle = \left(1 - i\int_0^t dt_1 \widetilde{H}_{2s}(t_1)\right)|\widetilde{\psi(0)}\rangle. \tag{18}$$

For example, if $|\widetilde{\psi(0)}\rangle$ is the vacuum state of string $|0\rangle$ then

$$\begin{aligned}
|\widetilde{\psi(t)}\rangle &= |0\rangle + i\sum_{n,l,s} \frac{M_{n,l,s}}{\sqrt{(2L)^4\omega_n\omega_l\omega_s\omega_{n+l+s}}}|n,l,s,-n-l-s\rangle \\
&\quad \times \frac{\sin\frac{(\omega_n+\omega_l+\omega_s+\omega_{n+l+s})}{2}t}{\frac{\omega_n+\omega_l+\omega_s+\omega_{n+l+s}}{2}} e^{i\frac{(\omega_n+\omega_l+\omega_s+\omega_{n+l+s})t}{2}},
\end{aligned} \tag{19}$$

and If $|\widetilde{\psi(0)}\rangle = |j\rangle$, i.e, the string is in its j-th mode, for some j then

$$|\widetilde{\psi(t)}\rangle = |j\rangle + 4i \sum_{n,s} \frac{M_{n,s,-j}}{\sqrt{(2L)^4 \omega_n \omega_s \omega_j \omega_{j-n-s}}} |n, s, j-n-s\rangle$$
$$\times \frac{\sin\left(\frac{(\omega_n+\omega_s+\omega_{j-n-s}-\omega_j)}{2}\right) t}{\frac{(\omega_n+\omega_s+\omega_{j-n-s}-\omega_j)}{2}} e^{i\frac{(\omega_n+\omega_s+\omega_{j-n-s}-\omega_j)t}{2}}$$
$$+ i \sum_{n,l,s} \frac{M_{n,l,s}}{\sqrt{(2L)^4 \omega_n \omega_l \omega_s \omega_{n+l+s}}} |j, n, l, s, -n-l-s\rangle$$
$$\times \frac{\sin\left(\frac{(\omega_n+\omega_l+\omega_s+\omega_{n+l+s})}{2}\right) t}{\frac{(\omega_n+\omega_l+\omega_s+\omega_{n+l+s})}{2}} e^{i\frac{(\omega_n+\omega_l+\omega_s+\omega_{n+l+s})t}{2}}, \tag{20}$$

which gives the probability of transition $|j\rangle \to |p, q, r\rangle$ after passing a long time

$$|\langle p, q, r|\widetilde{\psi(t)}\rangle|^2 = \frac{32\pi^9 \gamma^2 t}{L^6 \omega_p \omega_q \omega_r \omega_j} (jpqr)^2 \delta_{j,p+q+r} \delta$$
$$\times (\omega_p + \omega_q + \omega_r - \omega_j) \tag{21}$$

We can also determine $C_n(t)$ up to the first-order approximation to be

$$C_n(t) = E_n e^{i\omega_n t} + F_n e^{-i\omega_n t}$$
$$+ \sum_{s,j} \frac{L_{-s,-j} B_s^\dagger B_j^\dagger B_{s+j+n}}{\omega_n^2 - (\omega_s + \omega_j - \omega_{s+j-n})^2} e^{i(\omega_s+\omega_j-\omega_{s+j-n})t}$$
$$+ \sum_{s,j} \frac{L_{-s,-j} B_s^\dagger B_j^\dagger B_{-s-j-n}}{\omega_n^2 - (\omega_s + \omega_j + \omega_{s+j-n})^2} e^{i(\omega_s+\omega_j+\omega_{s+j-n})t}$$
$$+ \sum_{s,j} \frac{L_{-s,j} B_s^\dagger B_j B_{s+n-j}}{\omega_n^2 - (\omega_s - \omega_j - \omega_{s+j-n})^2} e^{i(\omega_s-\omega_j-\omega_{s+j-n})t}$$
$$+ \sum_{s,j} \frac{L_{-s,j} B_s^\dagger B_{j-s-n}^\dagger B_j}{\omega_n^2 - (\omega_s - \omega_j + \omega_{s+j-n})^2} e^{i(\omega_s-\omega_j+\omega_{s+j-n})t}$$
$$+ \sum_{s,j} \frac{L_{s,-j} B_j^\dagger B_s B_{n+j-s}}{\omega_n^2 - (\omega_j - \omega_s - \omega_{s+j-n})^2} e^{i(\omega_j-\omega_s-\omega_{s+j-n})t}$$
$$+ \sum_{s,j} \frac{L_{s,-j} B_j^\dagger B_{s-n-j}^\dagger B_s}{\omega_n^2 - (\omega_j - \omega_s + \omega_{s+j-n})^2} e^{i(\omega_j-\omega_s+\omega_{s+j-n})t}$$

$$+\sum_{s,j}\frac{L_{s,j}B_jB_sB_{n-s-j}}{\omega_n^2-(\omega_j+\omega_s+\omega_{s+j-n})^2}e^{-i(\omega_j+\omega_s+\omega_{s+j-n})t}$$

$$+\sum_{s,j}\frac{L_{s,j}B^{\dagger}_{s+j-n}B_jB_s}{\omega_n^2-(\omega_j+\omega_s-\omega_{s+j-n})^2}e^{-i(\omega_j+\omega_s-\omega_{s+j-n})t} \tag{22}$$

and the operators E_n, F_n, can be determined from the initial conditions

$$C_n(0)=B^{\dagger}_{-n}+B_n,$$
$$\dot{C}_n(0)=i\omega_nB^{\dagger}_{-n}-i\omega_nB_n. \tag{23}$$

3. INTERACTION WITH THE QUANTUM VACUUM FIELD

When the quantized vibrating string interacts with the quantum vacuum of the electromagnetic field, the total Hamiltonian can be written as

$$H=H_s+H_{\mathrm{F}}+H'=H_{1s}+H_{2s}+H_{\mathrm{F}}+H', \tag{24}$$

where $H_s=H_{1s}+H_{2s}$, is the string Hamiltonian defined in (9), H_{F} is the Hamiltonian of the vacuum field and H' is the interaction part, defined by

$$H_{\mathrm{F}}=\int d^3x\left[-\frac{1}{2}\pi_\mu\pi^\mu+\frac{1}{2}\sum_{k=1}^{3}(\partial_kA_\nu)(\partial^kA^\nu)\right]$$
$$H'=\int d^3xj_\mu A^\mu, \tag{25}$$

where $\pi^\mu=-\frac{\partial A^\mu}{\partial t}$ is canonical momentum density corresponding to A^μ and the Lagrangian density of the electromagnetic field is (Greiner, 1996)

$$\pounds'=-\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)-j_\mu A^\mu. \tag{26}$$

Suppose a vibrating medium has the electrical charge density $\rho(x)$, and let $\vec{\eta}(\vec{x},t)$ be the mechanical wave propagating in the medium, i.e, $\vec{\eta}(\vec{x},t)$ is the departure from the stable state of an infinitesimal element with center $\vec{x}$, this vibrating medium interacts with the quantized electromagnetic field as follows,

$$H'=\int j_\mu(\vec{x},t)A^\mu(\vec{x},t)d^3x=\int d^3x\rho(\vec{x})\Bigg[A^0(\vec{x},t)+\vec{\nabla}A^0(\vec{x},t)\cdot\vec{\eta}(\vec{x},t)$$
$$-\frac{\partial\vec{\eta}(\vec{x},t)}{\partial t}\cdot\vec{A}(\vec{x},t)-(\vec{\eta}(\vec{x},t)\cdot\nabla)\vec{A}(\vec{x},t)\cdot\frac{\partial\vec{\eta}(\vec{x},t)}{\partial t}+\cdots\Bigg], \tag{27}$$

where $\rho(\vec{x})$ is the charge density before the medium starts vibrating. We rewrite H' up to the dipole approximation so

$$\int j_\mu(\vec{x},t)A^\mu(\vec{x},t)d^3x \sim \int d^3x\rho(\vec{x})(\vec{\nabla}A^0(\vec{x},t) + \frac{\partial\vec{A}(\vec{x},t)}{\partial t})\cdot\vec{\eta}(\vec{x},t) = -\int d^3x\vec{P}(\vec{x},t)\cdot\vec{E}(\vec{x},t), \tag{28}$$

where $\vec{P}(\vec{x},t)$ is the electric dipole density of the vibrating medium. For the string with a charge density σ and length L stretched on the x^1 axis with its two ends at $x=0$ and $x=L$, we can write the charge density field as follows

$$\rho(\vec{x}) = \sigma\delta(x^2)\delta(x^3)(u(x^1) - u(x^1 - L)), \tag{29}$$

where u is the step function, assume that the string vibrates only in the x^2 direction such that $y(x^1,t)\vec{j}$ be the corresponding mechanical wave function, so by using (28), the interaction part of the Hamiltonian can be written as

$$H'(t) = -\sigma\int_0^L dx^1 y(x^1,t)E^2(x^1,0,0,t) \tag{30}$$

By defining $H_0 := H_{1s} + H_{\mathrm{F}}$ and $H'' := H_{2s} + H'$ such that $H = H_0 + H''$, we have in the interaction picture

$$y_I(x^1,t) = e^{iH_0t}y(x^1,0)e^{-iH_0t} = \sum_{n=-\infty}^{+\infty}\left(B_ne^{-i|\omega_n|t+ik_nx^1} + B_n^\dagger e^{i|\omega_n|t-ik_nx^1}\right)$$

$$(\pi_y)_I(x^1,t) = e^{iH_0t}\pi_y(x^1,0)e^{-iH_0t} = i\sum_{n=-\infty}^{+\infty}|\omega_n|\left(B_n^\dagger e^{i|\omega_n|t-ik_nx^1} - B_ne^{-i|\omega_n|t+ik_nx^1}\right) \tag{31}$$

where $k_n = \frac{2\pi n}{L}$, $\omega_n = \frac{2\pi n v}{L}$. The new annihilation and creation operators B_n and $B_m^\dagger$ of the string satisfy the commutation relation (13). For electromagnetic field we can write the following expansions in the interaction picture

$$A_I^\mu(\vec{x},t) = \int\frac{d^3k}{\sqrt{2(2\pi)^3\omega_k}}\sum_{\lambda=0}^{3}(b_{k\lambda}^\dagger e^{i\omega_kt-i\vec{k}\cdot\vec{x}} + b_{k\lambda}e^{-i\omega_kt+i\vec{k}\cdot\vec{x}})\varepsilon^\mu(\vec{k},\lambda),$$

$$\pi_I^\mu(\vec{x},t) = -\int\frac{d^3ki\omega_k}{\sqrt{2(2\pi)^3\omega_k}}\sum_{\lambda=0}^{3}\left(b_{k\lambda}^\dagger e^{i\omega_kt-i\vec{k}\cdot\vec{x}} - b_{k\lambda}e^{-i\omega_kt+i\vec{k}\cdot\vec{x}}\right)\varepsilon^\mu(\vec{k},\lambda),$$

$$\vec{E}_I(\vec{x},t) = i\sum_{\lambda=1}^{2}\int d^3k\sqrt{\frac{\omega_k}{2(2\pi)^3}}\vec{\varepsilon}(\vec{k},\lambda)\left(b_{k\lambda}e^{-i\omega_kt+i\vec{k}\cdot\vec{x}} - b_{k\lambda}^\dagger e^{i\omega_kt-i\vec{k}\cdot\vec{x}}\right), \tag{32}$$

and the creation and annihilation operators $b_{k\lambda}$ and $b^{\dagger}_{k\lambda}$ satisfy the computation relation

$$[b_{k\lambda}, b^{\dagger}_{k'\lambda'}] = \delta_{\lambda\lambda'}\delta(\vec{k} - \vec{k}'), \tag{33}$$

one can easily obtain the Hamiltonian $(H_0)_I$ in the interaction picture

$$(H_0)_I = H_0 = L \sum_{n=-\infty}^{+\infty} \omega_n^2 B_n^{\dagger} B_n + \int d^3k \sum_{\lambda=1}^{2} \omega_k b^{\dagger}_{k\lambda} b_{k\lambda}. \tag{34}$$

If we ignore H_{2s} in the Hamiltonian H'', i.e, $H'' = H'$, and write H' in the interaction picture as

$$H'_I(t) = -\sigma \int_0^L y_I(x^1, t) E_I^2(x^1, 0, 0, t) dx^1, \tag{35}$$

then by substituting $y_I(x^1, t)$ from (31) and $E_I^2(x^1, 0, 0, t)$ from (32) one can obtain $H'_I(t)$ and calculate the probability amplitude for various transitions. Up to the first-order perturbation, the evolution operator $U^{(1)}$, in interaction picture, is

$$U^{(1)}(+\infty, -\infty) = 1 - i \int_{-\infty}^{+\infty} H'_I(t) dt, \tag{36}$$

for example, the probability amplitude for transition from $|0\rangle_F \otimes |m\rangle_s$ to
$|\vec{q}, r\rangle_F \otimes |0\rangle_s$
is

$$\frac{-i\sigma\sqrt{\omega_q}}{2\sqrt{2\pi L|\omega_m|}} \varepsilon^2(\vec{q}, r)\delta(\omega_q - |\omega_m|) \frac{e^{-iq^1 L} - 1}{k_m - q^1}, \tag{37}$$

where $\vec{q}$ and r are momentum and polarization of the created photon respectively, m is the quantum number for string quanta, $|0\rangle_F$ and $|0\rangle_s$ are electromagnetic and string vacuums respectively. In the second-order perturbation we have

$$\begin{aligned} U^{(2)}(+\infty, -\infty) = 1 - i \int_{-\infty}^{+\infty} H'_I(t) dt \\ - \frac{1}{2} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 T(H'(t_1) H'(t_2)), \end{aligned} \tag{38}$$

by using of wick theorem (Greiner, 1996) the last term can be written as

$$\begin{aligned} & -\frac{1}{2} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 T(H'(t_1) H'(t_2)) \\ & = -\frac{\sigma^2}{2} \int_{-\infty}^{+\infty} dt_1 \int_0^L dx^1 \int_{-\infty}^{+\infty} dt_2 \int_0^L dz^1 \\ & \{: y_I(x^1, t_1) E_I^2(x^1, 0, 0, t_1) y_I(z^1, t_2) E_I^2(z^1, 0, 0, t_2) : \end{aligned}$$

$$+ : E_I^2(x^1, 0, 0, t_1)E_I^2(z^1, 0, 0, t_2) : \langle 0|y_I(x^1, t_1)y_I(z^1, t_2)|0\rangle$$
$$+ : y_I(x^1, t_1)y_I(z^1, t_2) : \langle 0|E_I^2(x^1, 0, 0, t_1)E_I^2(z^1, 0, 0, t_2)|0\rangle$$
$$+ \langle 0|y_I(x^1, t_1)y_I(z^1, t_2)|0\rangle\langle 0|E_I^2(x^1, 0, 0, t_1)E_I^2(z^1, 0, 0, t_2)|0\rangle\big\}. \quad (39)$$

where : : denote the normal ordering. The second term under integral has no effect on the probability amplitude of those transitions that initial and final string states are different. The third term under integral has no effect on the probability amplitude of those transitions that initial and final photon states are different, also the last term has no effect on the probability amplitude of those transitions that initial and final states both string and electromagnetic field are different. By substituting $y_I(x^1, t)$ from (31) and $E_I^3(x^1, 0, 0, t)$ from (32) into the first term in above equation, one can calculate the probability amplitude for some special transitions. For example, probability amplitude for the transition $|m\rangle \otimes |\vec{p}, r_1\rangle_F \rightarrow |n\rangle \otimes |\vec{q}, r_2\rangle_F$ is

$$-\frac{\sigma^2\sqrt{\omega_q\omega_p}}{16\pi L\sqrt{|\omega_m||\omega_n|}}\varepsilon^2(\vec{p}, r_1)\varepsilon^2(\vec{q}, r_2)\delta(\omega_q - |\omega_m|)$$
$$\times\, \delta(\omega_p - |\omega_n|)\frac{(e^{-iq^1L} - 1)(e^{ip^1L} - 1)}{(k_n - p^1)(k_m - q^1)}, \quad (40)$$

If we keep H_{2s} in H'', then $H'' = H' + H_{2s}$ and up to the first-order perturbation, we have

$$U^{(1)}(+\infty, -\infty) = 1 - i\int_{-\infty}^{+\infty} dt \int_0^L dx^1 \left[\frac{\gamma}{24}\left(\frac{\partial y_I(x^1, t)}{\partial x^1}\right)^4 + \sigma y_I(x^1, t)E_I^2(x^1, 0, 0, t)\right], \quad (41)$$

in this case, the term $\frac{1}{24}(\frac{\partial y(x^1,t)}{\partial x^1})^4$ has no effect on the probability amplitude of those transitions that the initial and the final photon states are different. In the second-order perturbation, we can write

$$-\frac{1}{2}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 T(H'(t_1)H'(t_2))$$
$$= -\frac{1}{2}\int_{-\infty}^{+\infty} dt_1 \int_0^L dx^1 \int_{-\infty}^{+\infty} dt_2 \int_0^L dz^1$$
$$\times \left\{\frac{\gamma^2}{576}T\left[:\left(\frac{\partial y(x^1, t_1)}{\partial x^1}\right)^4 :: \left(\frac{\partial y(z^1, t_2)}{\partial z^1}\right)^4 :\right]\right.$$
$$+ \frac{\gamma\sigma}{24}T\left[:\left(\frac{\partial y(x^1, t_1)}{\partial x^1}\right)^4 : y(z^1, t_2)E^2(z^1, 0, 0, t_2) :\right]$$

$$+\frac{\gamma\sigma}{24}T\left[:y(x^1,t_1)E^2(x^1,0,0,t_1)::\left(\frac{\partial y(z^1,t_2)}{\partial z^1}\right)^4\right]$$

$$\left.-\sigma^2 T[:y(x^1,t_1)E^2(x^1,0,0,t_1)::y(z^1,t_2)E^2(z^1,0,0,t_2):]\right\}. \qquad (42)$$

The first term under integral has no effect on those probability amplitude that the initial photon state is different from final photon state. The effect of fourth term under integral is similar to (39) and has no effect on some of transitions. For example, transition $|m\rangle_s \otimes |l\rangle_s |f\rangle_s \otimes |0\rangle_F \longrightarrow |n\rangle_s \otimes |j\rangle_s \otimes |\vec{p},r\rangle_F$ up to the second-order perturbation, has the following probability amplitude

$$\frac{-\gamma\sigma\varepsilon^3(\vec{p},r)}{4\sqrt{\omega_l\omega_f\omega_m\omega_n\omega_j}}\sqrt{\frac{(2\pi)^9\omega_p}{L^{11}}}$$

$$\times\left\{mlnj\delta(\omega_n+\omega_j-\omega_m-\omega_l)\delta(\omega_f-\omega_p)\delta_{m+l,n+j}\frac{e^{-ip^1L}-1}{p^1-\frac{2\pi f}{L}}\right.$$

$$+mfnj\delta(\omega_n+\omega_j-\omega_m-\omega_f)\delta(\omega_l-\omega_p)\delta_{m+f,n+j}\frac{e^{-ip^1L}-1}{p^1-\frac{2\pi l}{L}}$$

$$\left.+lfnj\delta(\omega_n+\omega_j-\omega_l-\omega_f)\delta(\omega_m-\omega_p)\delta_{l+f,n+j}\frac{e^{-ip^1L}-1}{p^1-\frac{2\pi m}{L})}\right\}, \qquad (43)$$

which is only the effect of the second and third term in (42). That is only the non-linear term $\frac{\gamma}{24}(\frac{\partial y}{\partial x})^4$ in Hamiltonian (5) has a non-zero effect in this probability. Also the probability amplitude for transition $|m\rangle_s \otimes |l\rangle_s \otimes |f\rangle_s \otimes |j\rangle_s \otimes |0\rangle_F \longrightarrow |n\rangle_s \otimes |\vec{p},r\rangle_F$ is

$$\frac{-\gamma\sigma\varepsilon^3(\vec{p},r)}{4\sqrt{\omega_l\omega_f\omega_m\omega_n\omega_j}}\sqrt{\frac{(2\pi)^9\omega_p}{L^{11}}}$$

$$\times\left\{mlnf\delta(\omega_n-\omega_f-\omega_m-\omega_l)\delta(\omega_j-\omega_p)\delta_{m+l+f,n}\frac{e^{-ip^1L}-1}{p^1-\frac{2\pi j}{L}}\right.$$

$$+(mlnj\delta(\omega_n-\omega_l-\omega_m-\omega_j)\delta(\omega_f-\omega_p)\delta_{m+l+j,f}\frac{e^{-ip^1L}-1}{p^1-\frac{2\pi f}{L}}$$

$$+lfnj\delta(\omega_n-\omega_j-\omega_l-\omega_f)\delta(\omega_m-\omega_p)\delta_{l+f+j,n}\frac{e^{-ip^1L}-1}{p^1-\frac{2\pi m}{L}}$$

$$\left.+mfnj\delta(\omega_n-\omega_j-\omega_m-\omega_f)\delta(\omega_l-\omega_p)\delta_{m+f+j,n}\frac{e^{-ip^1L}-1}{p^1-\frac{2\pi l}{L}}\right\} \qquad (44)$$

which is only due to the effect of non-linear term $\frac{\gamma}{24}(\frac{\partial y}{\partial x^1})^4$

4. THE EFFECT OF VACUUM FIELD AND RADIATION REACTION

In this section, we use the Coulomb gauge for finding the radiation reaction effect. For this purpose, let us write the free part of the electromagnetic field as

$$H_{\rm F} = \frac{1}{2}\int d^3x((\vec{E}^{\parallel})^2 + (\vec{E}^{\perp})^2 + 2\vec{E}^{\perp}.\vec{E}^{\parallel} + \vec{B}^2) \tag{45}$$

where $\vec{E}^{\perp} = -\frac{\partial\vec{A}}{\partial t}$ and $\vec{E}^{\parallel} = -\vec{\nabla}A^0$, $H_{\rm F}$ can be rewritten simply as

$$H_{\rm F} = \frac{1}{2}\int d^3x((\vec{E}^{\perp})^2 + \vec{B}^2) + \frac{1}{2}\int d^3x j_0(\vec{x},t)A^0(\vec{x},t), \tag{46}$$

where the last term is the Coulomb interaction of the string with itself which clearly is divergent and we may ignore it because we are not interested in self interaction effects. So the effective Hamiltonian can be considered to be

$$H = H_{\rm F} = \frac{1}{2}\int d^3x((\vec{E}^{\perp})^2 + \vec{B}^2) = \sum_{\lambda=1}^{2}\omega_k\left(a^{\dagger}_{k\lambda}(t)a_{k\lambda}(t) + \frac{1}{2}\right), \tag{47}$$

and accordingly the total Hamiltonian is

$$\begin{aligned} H &= H_s + H_{\rm F} + H' \\ &= H_s + \sum_{\lambda=1}^{2}\omega_k\left(a^{\dagger}_{k\lambda}(t)a_{k\lambda}(t) + \frac{1}{2}\right) - \int dx^3\,\vec{j}(\vec{x},t).\vec{A}(\vec{x},t), \end{aligned} \tag{48}$$

where H_s is defined in (5). The interaction part of the Hamiltonian (27) up to the electric dipole approximation is given by (30) which gives the Heisenberg equation for $y(x^1, t)$ as

$$\frac{\partial^2 y}{\partial t^2} - v^2\frac{\partial^2 y}{\partial(x^1)^2} = \frac{\gamma}{2}\left(\frac{\partial y}{\partial x^1}\right)^2\frac{\partial^2 y}{\partial(x^1)^2} + \sigma E^2(x^1, 0, 0, t). \tag{49}$$

The vector potential $\vec{A}$ and transverse electrical field are defined by (Milonni, 1994)

$$\begin{aligned} \vec{A}(\vec{x},t) &= \int\frac{d^3k}{\sqrt{2(2\pi)^3\omega_k}}\sum_{\lambda=1}^{2}(a_{k\lambda}(t)e^{+i\vec{k}.\vec{x}} + a^{\dagger}_{k\lambda}(t)e^{-i\vec{k}.\vec{x}})\vec{\varepsilon}(\vec{k},\lambda) \\ \vec{E}^{\perp}(\vec{x},t) &= \int\frac{d^3ki\omega_k}{\sqrt{2(2\pi)^3\omega_k}}\sum_{\lambda=1}^{2}(a_{k\lambda}(t)e^{+i\vec{k}.\vec{x}} - a^{\dagger}_{k\lambda}(t)e^{-i\vec{k}.\vec{x}})\vec{\varepsilon}(\vec{k},\lambda), \end{aligned} \tag{50}$$

where the time dependence of $a_{k\lambda}(t)$ is not simply as $a_{k\lambda}e^{-i\omega_k t}$ and must be specified by Heisenberg equation so

$$\begin{aligned}\dot{a}_{k\lambda}(t) &= i[H, a_{k\lambda}(t)] = i[H_s + H_{\mathrm{F}} + H'] \\ &= -i\omega_k a_{k\lambda}(t) - \sigma\sqrt{\frac{\omega_k}{2(2\pi)^3}}\varepsilon^2(\vec{k},\lambda)\int_0^L dz y(z,t)e^{-ik^1 z},\end{aligned} \tag{51}$$

where we have used the commutation relations $[a_{k'\lambda'}(t), a^{\dagger}_{k\lambda}(t)] = \delta_{\lambda\lambda'}\delta(\vec{k}-\vec{k}')$. A formal solution for above equation can be written as

$$\begin{aligned}a_{k\lambda}(t) &= a_{k\lambda}(0)e^{-i\omega_k t} \\ &\quad - \sigma\sqrt{\frac{\omega_k}{2(2\pi)^3}}\varepsilon^2(\vec{k},\lambda)\int_0^t dt' e^{i\omega_k(t-t')}\int_0^L dz y(z,t')e^{-ik^1 z},\end{aligned} \tag{52}$$

now by substituting $a_{k\lambda}(t)$ from Eq. (52) in electrical field expression (50), one obtains the electrical field as the sum of two parts, the first part is nothing but the vacuum field which is

$$\begin{aligned}E_0^2(x^1,0,0,t) &= i\int d^3k\sqrt{\frac{\omega_k}{2(2\pi)^3}}\sum_{\lambda=1}^{2}(a_{k\lambda}(0)e^{-i\omega_k t + ik^1x^1} \\ &\quad - a^{\dagger}_{k\lambda}(0)e^{i\omega_k t - ik^1 x^1})\varepsilon^2(\vec{k}\lambda),\end{aligned} \tag{53}$$

the second part is the radiation reaction field

$$\begin{aligned}E_{\mathrm{RR}}^2(x^1,0,0,t) &= -\frac{\sigma\pi}{(2\pi)^3}\int_0^{2\varphi} d\varphi\int_0^{\pi} d\theta\,\sin\theta\sum_{\lambda=1}^{2}(\varepsilon^2(\vec{k},\lambda))^2 \\ &\quad\times\int_0^t dt'\int_0^L dz y(z,t')\frac{\partial^3}{\partial(t')^3}\delta(t-t'+(x^1-z)\cos\theta).\end{aligned} \tag{54}$$

Integration by parts respect to t' and then inserting $\sum_{\lambda=1}^{2}(\varepsilon^2(\vec{k},\lambda))^2 = 1 - \sin^2\theta\cos^2\varphi$ and doing integrals with respect to θ, φ, we at last come to the following relation for radiation reaction component

$$E_{\mathrm{RR}}^2(x^1,0,0,t) = \frac{\sigma}{\pi}\int_0^L dz\sum_{m=0}^{\infty}\frac{(m+1)(x^1-z)^{2m}}{(2m)!(2m+1)(2m+3)}\frac{\partial^{2m+3}}{\partial t^{2m+3}}y(z,t), \tag{55}$$

the first term, i.e, $m = 0$ gives the first-order approximation of E_{RR}^2

$$E_{\mathrm{RR}}^2(x^1,0,0,t) = \frac{\sigma}{3\pi}\int_0^L dz\frac{\partial^3 y(z,t)}{\partial t^3}, \tag{56}$$

that may be compared with radiation reaction field due to one-dimensional harmonic oscillator (Milonni, 1994). Using (56), the Heisenberg equation (49) can be written like this

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial (x^1)^2} = \frac{\gamma}{24} \frac{\partial^2 y}{\partial (x^1)^2} \left(\frac{\partial y}{\partial x^1} \right)^2 + \frac{\sigma^2}{3\pi} \int_0^L dz \frac{\partial^3 y(z,t)}{\partial t^3} + \sigma E_0^2(x^1, 0, 0, t). \tag{57}$$

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